

Explicit expressions of spin wave functions

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We derive the explicit expressions of the canonical and helicity wave functions for massive particles with arbitrary spins. Properties of these wave functions are discussed.

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I. INTRODUCTION

To describe particles with high spins in amplitude analysis, one needs to construct the explicit expressions of wave functions. Detailed properties of the amplitudes are needed in tensor analysis [1–5] to give independent invariant amplitudes free of kinematics singularities and zeros [6–8]. We will give the explicit expressions of the canonical and helicity wave functions for massive particles with arbitrary spins in this paper. These wave functions satisfy Rarita-Schwinger conditions [9].

We will discuss quantum states in section II. Spin wave functions are given in section III and section IV.

II. QUANTUM STATES

Let $L(p)$ be a Lorentz transformation that satisfies

$$p^\mu = L^\mu{}_\nu(p)k^\nu. \quad (1)$$

For massive particles one can choose the standard momentum to be $(k^\mu) = (w; \vec{0})$. w is the mass of the particle. The space-time metric is taken as $(g^{\mu\nu}) = \text{diag}\{1, -1, -1, -1\}$. Now we can define one particle states as [10]

$$|p, \sigma\rangle = U(L(p))|k, \sigma\rangle \quad (2)$$

with $U(L(p))$ the unitary representation of $L(p)$ in Hilbert space. The one particle states satisfy

$$\hat{p}^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle. \quad (3)$$

We choose the orthonormal condition to be

$$\langle p', \sigma' | p, \sigma \rangle = (2\pi)^3 (2p^0) \delta(\vec{p}' - \vec{p}) \delta_{\sigma'\sigma}. \quad (4)$$

Under Lorentz transformations,

$$U(\Lambda)|p, \sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))|\Lambda p, \sigma'\rangle; \quad (5)$$

where

$$D_{\sigma'\sigma}(W(\Lambda, p)) \equiv L^{-1}(\Lambda p)\Lambda L(p) \quad (6)$$

is the Wigner rotation [11] and $\{D_{\sigma'\sigma}\}$ furnishes a representation of $SO(3)$. We also use the notation $|\vec{p}, j, m\rangle \equiv |p, j, m\rangle$.

There are infinite ways to define the Lorentz transformation that satisfies equation (1). Canonical state and helicity state are the two types mostly used.

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If one define the Lorentz transformation in equation (2) to be pure Lorentz boost

$$\begin{aligned} L(p) &= L(\vec{p}) \\ &\equiv R(\varphi, \theta, 0) L_z(|\vec{p}|) R^{-1}(\varphi, \theta, 0), \end{aligned} \quad (7)$$

the canonical state is obtained. Here $R(\varphi, \theta, 0)$ is the rotation that takes z -axis to the direction of \vec{p} with spherical angles (θ, φ) , and the boost $L_z(|\vec{p}|)$ takes the four-momentum $(k^\mu) = (w; \vec{0})$ to $(\sqrt{w^2 + \vec{p}^2}; 0, 0, |\vec{p}|)$. For a particle of spin- $j, \sigma \sim (j, m)$. It can be shown that for the canonical states, equation (7) become

$$U(\Lambda)|\vec{p}, j, m\rangle = \sum_{m'} D_{m' m}^j(L^{-1}(\vec{\Lambda}p)\Lambda L(\vec{p}))|\vec{\Lambda}p, j, m'\rangle. \quad (8)$$

$D_{m' m}^j$ is the ordinary D -function. Especially, under rotation R ,

$$U(R)|\vec{p}, j, m\rangle = \sum_{m'} D_{m' m}^j(R)|\vec{R}p, j, m'\rangle. \quad (9)$$

Defining the Lorentz transformation in another way will leads to helicity states [12]:

$$\begin{aligned} L(p) &= L(\vec{p})R^{-1}(\varphi, \theta, 0) \\ &\equiv R(\varphi, \theta, 0)L_z(|\vec{p}|). \end{aligned} \quad (10)$$

We have

$$U(\Lambda)|\vec{p}, j, \lambda\rangle = \sum_{\lambda'} D_{\lambda' \lambda}^j(L^{-1}(\vec{\Lambda}p)\Lambda L(\vec{p}))|\vec{\Lambda}p, j, \lambda'\rangle \quad (11)$$

and

$$U(R)|\vec{p}, j, \lambda\rangle = |\vec{R}p, j, \lambda\rangle. \quad (12)$$

The two types of definitions are related to each other by

$$|\vec{p}, j, \lambda\rangle_{\text{helicity}} = \sum_m D_{m \lambda}^j(\varphi, \theta, 0)|\vec{p}, j, m\rangle_{\text{canonical}}. \quad (13)$$

We see that the definition of state depends on the choice of Lorentz transformation in equation (1). There *is* a definition called spinor state [13], which is different from that of equation (2) and does not depend on the specific choice of Lorentz transformation; but it makes things more complex and is seldom used.

Now we write quantum states in terms of creation and annihilation operators:

$$|\vec{p}, \sigma\rangle = \sqrt{(2\pi)^3 2p^0} a^\dagger(\vec{p}, \sigma)|0\rangle, \quad (14)$$

with $|0\rangle$ the vacuum state. Quantum fields are given by [10]

$$\psi_l^{(+)} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_\sigma U_l(\vec{p}, \sigma) a(\vec{p}, \sigma) e^{-ip \cdot x}, \quad (15)$$

$$\psi_l^{(-)} = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}} \sum_\sigma V_l(\vec{p}, \sigma) a^\dagger(\vec{p}, \sigma) e^{ip \cdot x}, \quad (16)$$

$$U(\Lambda, a) \psi_l^{(\pm)} U^{-1}(\Lambda, a) = \sum_{l'} G_{ll'}(\Lambda^{-1}) \psi_{l'}^{(\pm)}(\Lambda x + a). \quad (17)$$

The coefficient functions, U_l and V_l , are wave functions in momentum space. a^μ are parameters for translation. $\{G_{ll'}\}$ furnishes a representation of the Lorentz group. One finds that wave functions satisfy [10]

$$\sum_{l'} G_{ll'}(\Lambda) U_{l'}(\vec{p}, \sigma) = \sum_{\sigma'} D_{\sigma' \sigma}(W(\Lambda, p)) U_l(\vec{\Lambda}p, \sigma'), \quad (18)$$

$$\sum_{l'} G_{ll'}(\Lambda) V_{l'}(\vec{p}, \sigma) = \sum_{\sigma'} D_{\sigma' \sigma}^*(W(\Lambda, p)) V_l(\vec{\Lambda}p, \sigma'); \quad (19)$$

so we can define wave functions as

$$U_l(\vec{p}, \sigma) = \sum_{l'} G_{ll'}(L(\vec{p})) U_{l'}(\vec{k}, \sigma), \quad (20)$$

$$V_l(\vec{p}, \sigma) = \sum_{l'} G_{ll'}(L(\vec{p})) V_{l'}(\vec{k}, \sigma). \quad (21)$$

For massive particles, $\vec{k} = \vec{0}$.

III. WAVE FUNCTIONS FOR INTEGRAL SPIN PARTICLES

If the index l in previous section is chosen as Lorentz indexes, one arrived at vector fields:

$$\begin{aligned} G(\Lambda)^\mu{}_\nu &= \Lambda^\mu{}_\nu, \\ U^\mu(\vec{p}, \sigma) &= L(p)^\mu{}_\nu U^\nu(\vec{0}, \sigma), \\ V^\mu(\vec{p}, \sigma) &= L(p)^\mu{}_\nu V^\nu(\vec{0}, \sigma). \end{aligned} \quad (22)$$

We use the following infinitesimal generators of the Lorentz group:

$$\begin{aligned} (J_1^\mu{}_\nu) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (J_2^\mu{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (J_3^\mu{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ (K_1^\mu{}_\nu) &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K_2^\mu{}_\nu) = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K_3^\mu{}_\nu) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \end{aligned} \quad (23)$$

and get the explicit expressions of canonical wave functions (E is the energy of the particle)

$$\begin{aligned} (e_c^\mu(\vec{p}, 0)) &= \begin{pmatrix} \frac{|\vec{p}|}{w} \cos \theta \\ \frac{1}{2} \left(\frac{E}{w} - 1 \right) \sin 2\theta \cos \varphi \\ \frac{1}{2} \left(\frac{E}{w} - 1 \right) \sin 2\theta \sin \varphi \\ \frac{1}{2} \left(\frac{E}{w} - 1 \right) (1 + \cos 2\theta) + 1 \end{pmatrix}, \\ (e_c^\mu(\vec{p}, \pm 1)) &= \mp \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{|\vec{p}|}{w} \sin \theta e^{\pm i\varphi} \\ \left(\frac{E}{w} - 1 \right) \sin^2 \theta \cos \varphi e^{\pm i\varphi} + 1 \\ \left(\frac{E}{w} - 1 \right) \sin^2 \theta \sin \varphi e^{\pm i\varphi} \pm 1 \\ \left(\frac{E}{w} - 1 \right) \cos \theta \sin \theta e^{\pm i\varphi} \end{pmatrix}; \end{aligned} \quad (24)$$

while helicity wave functions are

$$\begin{aligned} (e_h^\mu(\vec{p}, 0)) &= \begin{pmatrix} \frac{|\vec{p}|}{w} \sin \theta \cos \varphi \\ \frac{E}{w} \sin \theta \sin \varphi \\ \frac{E}{w} \cos \theta \\ 0 \end{pmatrix}, \\ (e_h^\mu(\vec{p}, \pm 1)) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp \cos \theta \cos \varphi + i \sin \varphi \\ \mp \cos \theta \sin \varphi - i \cos \varphi \\ \pm \sin \theta \end{pmatrix}. \end{aligned} \quad (25)$$

We have

$$U(\vec{p}, \sigma) = V^*(\vec{p}, \sigma) = e(\vec{p}, \sigma). \quad (26)$$

Wave functions for higher integral spins can be defined recurrently by

$$e_{\mu_1\mu_2\cdots\mu_j}(\vec{p}, j, \sigma) = \sum_{\sigma'_{j-1}, \sigma_j} (j-1, \sigma'_{j-1}; 1, \sigma_j | j, \sigma) e_{\mu_1\mu_2\cdots\mu_{j-1}}(\vec{p}, j-1, \sigma'_{j-1}) e_{\mu_j}(\vec{p}, \sigma_j). \quad (27)$$

Using the C-G coefficient relation

$$\begin{aligned} & \sum_{\sigma'_3, \sigma'_4, \dots, \sigma'_n} (j_1, \sigma_1; j_2, \sigma_2 | j_1 + j_2, \sigma'_3) (j_1 + j_2, \sigma'_3; k_3, \sigma_3 | j_1 + j_2 + j_3, \sigma'_4) \cdots \\ & \times (j_1 + j_2 + \cdots + j_{n-1}, \sigma'_n; j_n, \sigma_n | j_1 + j_2 + \cdots + j_n, \sigma'_n + \sigma_n) \\ & = \left[\prod_{i=1}^n \frac{(2j_i)!}{(j_i + \sigma_i)!(j_i - \sigma_i)!} \right]^{\frac{1}{2}} \left\{ \frac{\left[\sum_{i=1}^n (j_i + \sigma_i) \right]! \left[\sum_{i=1}^n (j_i - \sigma_i) \right]!}{\left(2 \sum_{i=1}^n j_i \right)!} \right\}^{\frac{1}{2}}, \end{aligned} \quad (28)$$

we find

$$\begin{aligned} & e_{\mu_1\mu_2\cdots\mu_j}(\vec{p}, j, \sigma) \\ & = \sum_{\sigma_1, \sigma_2, \dots, \sigma_j} \left\{ \frac{2^j (j+\sigma)! (j-\sigma)!}{(2j)! \prod_{i=1}^j [(1+\sigma_i)!(1-\sigma_i)!]} \right\}^{\frac{1}{2}} \delta_{\sigma_1+\sigma_2+\cdots+\sigma_j, \sigma} e_{\mu_1}(\vec{p}, \sigma_1) e_{\mu_2}(\vec{p}, \sigma_2) \cdots e_{\mu_j}(\vec{p}, \sigma_j). \end{aligned} \quad (29)$$

It is easy to show

$$\Lambda^{\mu_1}_{\nu_1} \Lambda^{\mu_2}_{\nu_2} \cdots \Lambda^{\mu_j}_{\nu_j} e^{\nu_1\nu_2\cdots\nu_j}(\vec{p}, j, \sigma) = \sum_{\sigma'} D^j_{\sigma'\sigma}(W(\Lambda, \vec{p})) e^{\mu_1\mu_2\cdots\mu_j}(\vec{\Lambda}p, j, \sigma'). \quad (30)$$

$e_{\mu_1\mu_2\cdots\mu_j}(\vec{p}, j, \sigma)$ satisfies all of the Rarita-Schwinger conditions: space-like, symmetric and traceless.

IV. WAVE FUNCTIONS FOR HALF-INTEGRAL SPIN PARTICLES

The convention of γ -matrices used here follows that of Bjorken and Drell [14], so the generators of the Lorentz group are

$$\vec{J} = \frac{1}{2} \begin{pmatrix} \vec{\tau} & 0 \\ 0 & \vec{\tau} \end{pmatrix}, \quad \vec{K} = \frac{i}{2} \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix}; \quad (31)$$

with $\vec{\tau}$ Pauli matrixes.

The spin- $\frac{1}{2}$ canonical wave functions are (here $\alpha = \ln((E + |\vec{p}|)/w)$)

$$\begin{aligned} U_c(\vec{p}, \frac{1}{2}) &= \begin{pmatrix} \cosh \frac{\alpha}{2} \\ 0 \\ \cos \theta \sinh \frac{\alpha}{2} \\ \sin \theta e^{i\varphi} \sinh \frac{\alpha}{2} \end{pmatrix}, \quad U_c(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} 0 \\ \cosh \frac{\alpha}{2} \\ \sin \theta e^{-i\varphi} \sinh \frac{\alpha}{2} \\ -\cos \theta \sinh \frac{\alpha}{2} \end{pmatrix}; \\ V_c(\vec{p}, \frac{1}{2}) &= \begin{pmatrix} \sin \theta e^{-i\varphi} \sinh \frac{\alpha}{2} \\ -\cos \theta \sinh \frac{\alpha}{2} \\ 0 \\ \cosh \frac{\alpha}{2} \end{pmatrix}, \quad V_c(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\cos \theta \sinh \frac{\alpha}{2} \\ -\sin \theta e^{i\varphi} \sinh \frac{\alpha}{2} \\ -\cosh \frac{\alpha}{2} \\ 0 \end{pmatrix}; \end{aligned} \quad (32)$$

and the helicity wave functions are

$$\begin{aligned} U_h(\vec{p}, \frac{1}{2}) &= \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ \sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ \cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ \sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \end{pmatrix}, \quad U_h(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ \cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ \sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ -\cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \end{pmatrix}; \\ V_h(\vec{p}, \frac{1}{2}) &= \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ -\cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ -\sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ \cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \end{pmatrix}, \quad V_h(\vec{p}, -\frac{1}{2}) = \begin{pmatrix} -\cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ -\sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} \sinh \frac{\alpha}{2} \\ -\cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \\ -\sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} \cosh \frac{\alpha}{2} \end{pmatrix}. \end{aligned} \quad (33)$$

Spin- $n + \frac{1}{2}$ wave functions read

$$\begin{aligned}
& U_{\mu_1 \mu_2 \dots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) \\
&= \sum_{\sigma_1, \sigma_2, \dots, \sigma_{n+1}} \left\{ \frac{2^n (n + \frac{1}{2} + \sigma)! (n + \frac{1}{2} - \sigma)!}{(2n+1)! \prod_{i=1}^n [(1+\sigma_i)!(1-\sigma_i)!]} \right\}^{\frac{1}{2}} \delta_{\sigma_1 + \sigma_2 + \dots + \sigma_{n+1}, \sigma} \\
& \quad \times e_{\mu_1}(\vec{p}, \sigma_1) e_{\mu_2}(\vec{p}, \sigma_2) \dots e_{\mu_n}(\vec{p}, \sigma_n) U(\vec{p}, \sigma_{n+1});
\end{aligned} \tag{34}$$

$$\begin{aligned}
& V_{\mu_1 \mu_2 \dots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) \\
&= \sum_{\sigma_1, \sigma_2, \dots, \sigma_{n+1}} \left\{ \frac{2^n (n + \frac{1}{2} + \sigma)! (n + \frac{1}{2} - \sigma)!}{(2n+1)! \prod_{i=1}^n [(1+\sigma_i)!(1-\sigma_i)!]} \right\}^{\frac{1}{2}} \delta_{\sigma_1 + \sigma_2 + \dots + \sigma_{n+1}, \sigma} \\
& \quad \times e_{\mu_1}^*(\vec{p}, \sigma_1) e_{\mu_2}^*(\vec{p}, \sigma_2) \dots e_{\mu_n}^*(\vec{p}, \sigma_n) V(\vec{p}, \sigma_{n+1}).
\end{aligned} \tag{35}$$

They satisfy Dirac equations and Rarita-Schwinger conditions [9]; especially

$$\gamma^{\mu_k} U_{\mu_1 \mu_2 \dots \mu_k \dots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = 0, \tag{36}$$

$$\gamma^{\mu_k} V_{\mu_1 \mu_2 \dots \mu_k \dots \mu_n}(\vec{p}, n + \frac{1}{2}, \sigma) = 0. \tag{37}$$

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